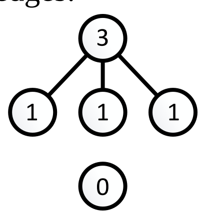
PART A – GRAPH THEORY – 25 MARKS

A1 Graphic Sequences (12 marks)

a) 3,3,1,1,0

This graph has 5 vertices.
It has a total degree of 8 and therefore 4 edges.

I can start by drawing the first vertex with 3 edges:

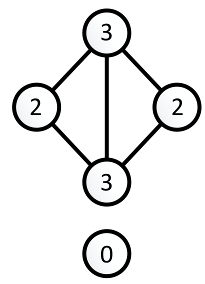


Only one edge has not been drawn and it is not enough to turn one of the vertices of degree 1 into a vertex of degree 3.
This sequence is not a graphic sequence.

b) 3,3,2,2,0

This graph has 5 vertices.
It has a total degree of 10 and therefore 5 edges.

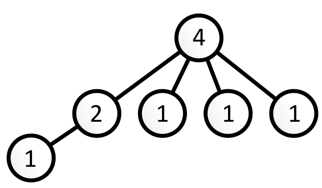
Here is a graph with this degree sequence:



c) 4,2,1,1,1,1

This graph has 6 vertices.
It has a total degree of 10 and therefore 5 edges.

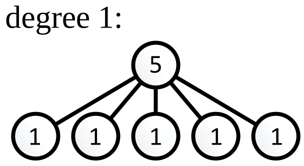
Here is a graph with this degree sequence:



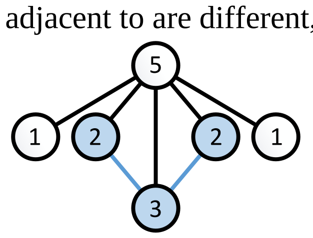
d) 5,3,3,1,1,1

This graph has 6 vertices.
It has a total degree of 14 and therefore 7 edges.

The first 5 edges are used to build a graph with a vertex of degree 5 and 5 vertices of degree 1:



The remaining two edges can be added to transform one of the vertices of degree 1 into a vertex of degree 3, but because the graph is simple, these two edges are not parallel and the other two vertices they are adjacent to are different, and of degree 2:



This sequence is not a degree sequence.

e) 5,3,3,2,2,1,1

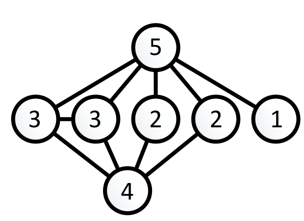
This graph has 7 vertices.
It has a total degree of 17 , which is an odd number.

Therefore by the handshake theorem, it is not possible to build such a graph.

f) 5,4,3,3,2,2,1

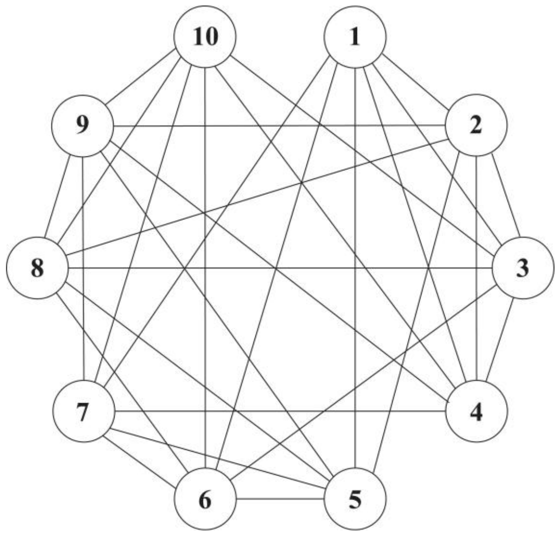
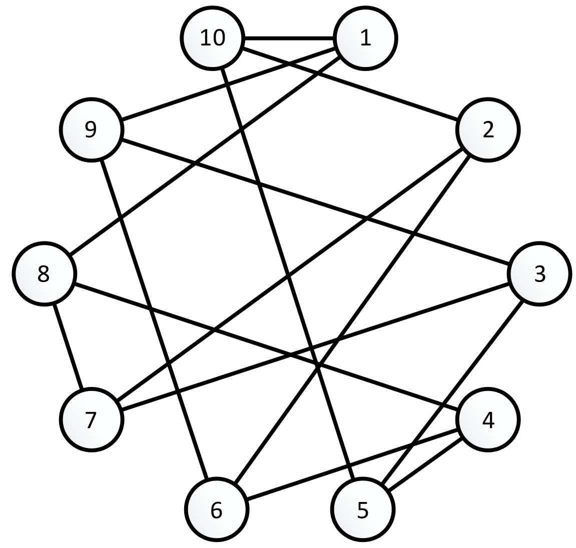
This graph has 7 vertices.
It has a total degree of 20 and therefore 10 edges.

Here is a graph with this degree sequence:

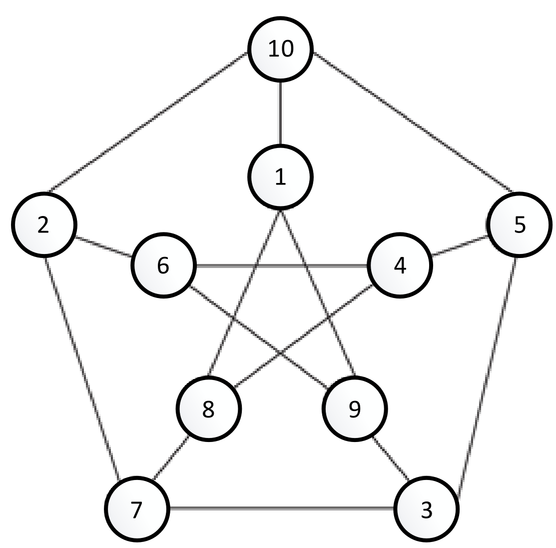


A2 Line and Petersen Graphs (13 marks)

a) (4 Marks)

 $L(K_5)$	 $\overline{L(K_5)} = (L(K_5))^C$

b) (4 marks) $\overline{L(K_5)}$, is isomorphic to the *Petersen graph* drawn below in more than one way. Identify one of these isomorphisms in the graph below, assuming that the isomorphic image of vertex 1 of $\overline{L(K_5)}$, is the vertex labelled 1 in the Petersen graph. Label the remaining vertices from 2 to 10 to describe this isomorphism.



c) (5 marks) In the two boxes below you are asked whether the Petersen graph contains a specific type of trail. If the answer is yes, give the trail; if not explain why not.

Does the Peterson graph contain an **Euler circuit**?	Does the Peterson graph contain a **Hamiltonian path**?
No because all the vertices are of odd degree.	Yes, for example: 1-9-6-4-8-7-2-10-5-3

PART B – SEQUENCES, RECURRENCE RELATIONS – 9 MARKS

Given the sequence a_n defined with the recurrence relation:

$a_0 = 0$
$a_k = 2a_{k-1} + 5k$ for $k \geq 1$

B1 Terms of the Sequence (5 marks)

$a_1 = 2.0 + 5.1 = 5.1$
$a_2 = 2(5.1) + 5.2 = 2.5.1 + 5.2$
$a_3 = 2(2.5.1 + 5.2) + 5.3 = 2^2.5.1 + 2^1.5.2 + 2^0.5.3$
$a_4 = 2(2^2.5.1 + 2^1.5.2 + 2^0.5.3) + 5.4 = 2^3.5.1 + 2^2.5.2 + 2^1.5.3 + 2^0.5.4$
$a_5 = 2(2^3.5.1 + 2^2.5.2 + 2^1.5.3 + 2^0.5.4)+ 5.5 = 2^4.5.1 + 2^3.5.2 + 2^2.5.3 + 2^1.5.4 + 2^0.5.5$
$= 5(2^4.1 + 2^3.2 + 2^2.3 + 2^1.4 + 2^0.5)$

B2 Iteration (4 marks)

$a_n = 5 \sum_{i=1}^{n} i.2^{n-i} = 5 \sum_{i=0}^{n-1} (n-i)2^i$

PART C – INDUCTION – 16 MARKS

This question works with the sequence a_n defined in part B. However, you do not need to finish part B in order to be able to do this question. In particular you cannot use the answer for B2 in this proof.

Prove by **mathematical (weak) induction** that for values of n strictly greater than 1, a_n is greater or equal to $5 \cdot 2^n$.
No other method is acceptable.
Be sure to lay out your proof clearly and correctly and to justify every step.

C1 Problem Statement (3 marks)

The conjecture that you are proving, is expressed symbolically in the form $\forall n \in D, P(n)$.

- $D = \mathbb{N} - \{0,1\}$
- $P(n)$ is: $a_n \geq 5 \cdot 2^n$

C2 Base Case (3 marks) Prove your base case here:

When n=2
$5.2^n = 5.2^2 = 5.4 = 20$
From B1, we have that $a_2 = 2.5.1 + 5.2 = 10+10 = 20$
Therefore $a_2 \geq 5 \cdot 2^2$ and therefore P(2) is true.

C3 Inductive step setup (4 marks)

- Assume that P(k) is true for some $k \geq 2$. I.e. $a_k \geq 5.2^k$ for some $k \geq 2$. This is the inductive hypothesis.
- We will show that P(k+1) is true, i.e. $a_{k+1} \geq 5.2^{k+1}$

C4 Remainder of Inductive step (6 marks).

Since $k \geq 2$, then some $k+1 \geq 3 \geq 1$, so the recursive definition of a applies to k+1:
$a_{k+1} = 2a_k + 5(k+1)$
$\geq 2(5.2^k)+5(k+1)$ by Inductive Hypothesis
$= 5.2^{k+1} + 5(k+1)$
$\geq 5.2^{k+1}$
QED